

# Secret Key and Private Key Constructions for Simple Multiterminal Source Models

Chunxuan Ye and Prakash Narayan

**Abstract**—We propose an approach for constructing secret and private keys based on the long-known Slepian-Wolf code, due to Wyner, for correlated sources connected by a virtual additive noise channel. Our work is motivated by results of Csiszár and Narayan which highlight innate connections between secrecy generation by multiple terminals that observe correlated source signals and Slepian-Wolf near-lossless data compression. Explicit procedures for such constructions and their substantiation are provided. The performance of low density parity check channel codes in devising a new class of secret keys is examined.

**Index terms:** Secret key construction, private key construction, secret key capacity, private key capacity, Slepian-Wolf data compression, binary symmetric channel, maximum likelihood decoding, LDPC codes.

## I. INTRODUCTION

The problem of secrecy generation by multiple terminals, based on their observations of separate but correlated signals followed by public communication among themselves, has been investigated by several authors ([23], [2], [7], among others). It has been shown that these terminals can generate secrecy, namely “common randomness” which is kept secret from an eavesdropper that is privy to said public communication and perhaps also to additional “wiretapped” side information.

Our work is motivated by [8] which studies secrecy generation for multiterminal “source models” with an arbitrary number of terminals, each of which observes a distinct component of a discrete memoryless multiple source (DMMS). Specifically, suppose that  $d \geq 2$  terminals observe, respectively,  $n$  independent and identically distributed (i.i.d.) repetitions of finite-valued random variables (rvs)  $X_1, \dots, X_d$ , denoted by  $\mathbf{X}_1, \dots, \mathbf{X}_d$ , where  $\mathbf{X}_i = (X_{i1}, \dots, X_{in})$ ,  $i = 1, \dots, d$ . Thereupon, unrestricted and noiseless public communication is allowed among the terminals. All such communication is observed by all the terminals and by the eavesdropper. The eavesdropper is assumed to be passive, i.e., unable to tamper with the public communication of the terminals. In

this framework, two models considered in [8] dealing with a *secret key* (SK) and a *private key* (PK) are pertinent to our work.

(i) *Secret key*: Suppose that all the terminals in  $\{1, \dots, d\}$  wish to generate a SK, i.e., common randomness which is concealed from the eavesdropper with access to their public communication and which is nearly uniformly distributed<sup>1</sup>. The largest (entropy) rate of such a SK, termed the SK capacity and denoted by  $C_S$ , is shown in [8] to equal

$$C_S = H(X_1, \dots, X_d) - R_{\min}, \quad (1)$$

where

$$R_{\min} = \min_{(R_1, \dots, R_d) \in \mathcal{R}} \sum_{i=1}^d R_i, \quad (2)$$

with<sup>2</sup>

$$\mathcal{R} = \{(R_1, \dots, R_d) : \sum_{i \in B} R_i \geq H(\{X_j, j \in B\} | \{X_j, j \in B^c\}), B \subset \{1, \dots, d\}\}, \quad (3)$$

where  $B^c = \{1, \dots, d\} \setminus B$ .

(ii) *Private key*: For a given subset  $A \subset \{1, \dots, d\}$ , a PK for the terminals in  $A$ , private from the terminals in  $A^c$ , is a SK generated by the terminals in  $A$  with the cooperation of the terminals in  $A^c$ , which is concealed from an eavesdropper with access to the public interterminal communication and also from the cooperating terminals in  $A^c$  (and, hence, private)<sup>3</sup>. The largest (entropy) rate of such a PK, termed the PK capacity and denoted by  $C_P(A)$ , is shown in [8] to be

$$C_P(A) = H(X_1, \dots, X_d) - H(\{X_i, i \in A^c\}) - R_{\min}(A) \\ = H(\{X_i, i \in A\} | \{X_i, i \in A^c\}) - R_{\min}(A), \quad (4)$$

where

$$R_{\min}(A) = \min_{\{R_i, i \in A\} \in \mathcal{R}(A)} \sum_{i \in A} R_i, \quad (5)$$

with

$$\mathcal{R}(A) = \{\{R_i, i \in A\} : \sum_{i \in B} R_i \geq H(\{X_j, j \in B\} | \{X_j, j \in B^c\}), B \subset A\}. \quad (6)$$

The expressions in (1)–(3) and (4)–(6) afford the following interpretation [8]. The joint entropy  $H(X_1, \dots, X_d)$  in (1)

<sup>1</sup>In [8], a general situation is studied in which a subset of the terminals generate a SK with the cooperation of the remaining terminals.

<sup>2</sup>Here,  $\subset$  denotes a proper subset.

<sup>3</sup>A general model is considered in [8] for privacy from a subset of  $A^c$  of the cooperating terminals.

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corresponds to the maximum rate of shared common randomness – sans secrecy constraints – that can ever be achieved by the terminals in  $\{1, \dots, d\}$  when each terminal becomes *omniscient*, i.e., reconstructs all the components of the DMMS with probability  $\cong 1$  as the observation length  $n$  becomes large. Further,  $R_{\min}$  in (2), (3) corresponds to the smallest aggregate rate of interterminal communication that enables every terminal to achieve omniscience [8]. Thus, from (1), the SK capacity  $C_S$ , i.e., largest rate at which all the terminals in  $\{1, \dots, d\}$  can generate a SK, is obtained by subtracting from the maximum rate of shared common randomness achievable by these terminals, viz.  $H(X_1, \dots, X_d)$ , the smallest overall rate  $R_{\min}$  of the (data-compressed) interterminal communication that enables all the terminals to become omniscient. A similar interpretation holds for the PK capacity  $C_P(A)$  in (4) as well, with the difference that the terminals in  $A^c$ , which cooperate in secrecy generation and yet must not be privy to the secrecy they help generate, can be assumed – without loss of generality – to simply “reveal” their observations [8]. Hence, the entropy terms in (1), (3) are now replaced in (4), (6) with additional conditioning on  $\{X_i, i \in A^c\}$ . It should be noted that  $R_{\min}$  and  $R_{\min}(A)$  are obtained as solutions to multiterminal Slepian-Wolf (SW) (near-lossless) data compression problems *not involving any secrecy constraints*.

The form of characterization of the SK and PK capacities in (1) and (4) also suggests successive steps for generating the corresponding keys. For instance, and loosely speaking, in order to generate a SK, the terminals in  $\{1, \dots, d\}$  first generate common randomness (without any secrecy restrictions) using SW-compressed interterminal communication denoted collectively by, say,  $\mathbf{F}$ . Thus, the terminals generate rvs  $L_i = L_i(\mathbf{X}_i, \mathbf{F})$ ,  $i \in \{1, \dots, d\}$ , with  $\frac{1}{n}H(L_i) > 0$ , which agree with probability  $\cong 1$  for  $n$  suitably large; suppressing subscripts, let  $L$  denote the resulting “common” rv where  $\frac{1}{n}H(L) > 0$ . The second step entails an extraction from  $L$  of a SK  $K = g(L)$  of entropy rate  $\frac{1}{n}H(L|\mathbf{F})$  by means of a suitable operation  $g$  performed *identically* at each terminal on the acquired common randomness  $L$ . In particular, when the common randomness acquired by the terminals corresponds to omniscience, i.e.,  $L \cong (\mathbf{X}_1, \dots, \mathbf{X}_d)$ , and is achieved using interterminal communication  $\mathbf{F}$  of the most parsimonious rate  $\cong R_{\min}$  in (2), then the corresponding SK  $K = g(L)$  has the best rate  $C_S$  given by (1). It is important to note, however, that as mentioned in ([8], Section VI) and already known from [23], [2], neither communication by every terminal nor omniscience is essential for generating secrecy (SK or PK) at the best rate; for instance, the rv  $L$  above need not correspond to omniscience for the SK  $K = g(L)$  to have the best possible rate in (1).

A similar approach as above can be used to generate a PK of the largest rate in (4).

The discussion above suggests that techniques for SW data compression could be used to devise constructive schemes for obtaining SKs and PKs that achieve the corresponding capacities. Further, in SW data compression, the existence of linear encoders of rates arbitrarily close to the SW bound has been long known [5]. In the special situation when the i.i.d. sequences observed at the terminals are related to each other in

probability law through virtual discrete memoryless channels (DMCs) characterized by independent additive noises, such linear SW encoders can be obtained in terms of cosets of linear error correction codes for such virtual channels, a fact first illustrated in [37] for the case of  $d = 2$  terminals connected by a virtual binary symmetric channel (BSC), and later exploited in most known linear constructions of SW encoders (cf. e.g., [1], [4], [11], [12], [15]–[17], [19], [20], [24], [29], [33]). When the i.i.d. sequences observed by  $d = 2$  terminals are connected by an arbitrary virtual DMC, the corresponding SW data compression can be viewed in terms of coding for a “semisymmetric” channel, i.e., a channel with independent additive noise that is defined over an enlarged alphabet [14]; the case of stationary ergodic observations at the terminals is also considered therein. These developments in SW data compression can translate into an emergence of new constructive schemes for secrecy generation.

Motivated by these considerations, we seek to devise new constructive schemes for secrecy generation in source models in which SW data compression plays a central role. The main technical contribution of this work is the following: Considering four simple models of secrecy generation, we show how a new class of SKs and PKs can be devised for them at rates arbitrarily close to the corresponding capacities, relying on the SW data compression code in [37]. Additionally, we examine the performance of low density parity check (LDPC) codes in the SW data compression step of the procedure for secrecy generation. Preliminary results of this work have been reported in [38], [39]. In independent work [25] for the case of  $d = 2$  terminals which is akin to but different from ours, extraction of a SK from previously acquired common randomness by means of a linear transformation has been demonstrated.

In related work, SK generation for a source model with two terminals that observe continuous-amplitude signals, has been studied in [40], [36], [26], [27], [41]. Furthermore, in recent years, several secrecy generation schemes have been reported, relying on capacity-achieving channel codes, for “wiretap” secrecy models that differ from ours. For instance, it was shown in [35] that such a channel code can attain the secrecy capacity for any wiretap channel. See also [3], [18].

The paper is organized as follows. Preliminaries are contained in Section II. In Section III, we consider four simple source models for which we provide elementary constructive schemes for SK or PK generation which rely on suitable SW data compression codes; the keys thereby generated are shown to satisfy the requisite secrecy and rate-optimality conditions in Section IV. Implementations of these constructions using LDPC codes are illustrated in Section V which also reports simulation results. Section VI contains closing remarks.

## II. PRELIMINARIES

### A. Secret Key and Private Key Capacities

Consider a DMMS with  $d \geq 2$  components, with corresponding generic rvs  $X_1, \dots, X_d$  taking values in finite alphabets  $\mathcal{X}_1, \dots, \mathcal{X}_d$ , respectively. Let  $\mathbf{X}_i = (X_{i,1}, \dots, X_{i,n})$  be  $n$  i.i.d. repetitions of rv  $X_i$ ,  $i \in \mathcal{D} = \{1, \dots, d\}$ . Terminals  $1, \dots, d$ , with respective observations  $\mathbf{X}_1, \dots, \mathbf{X}_d$ ,

represent the  $d$  users that wish to generate a SK by means of public communication. These terminals can communicate with each other through broadcasts over a noiseless public channel, possibly interactively in many rounds. In general, a communication from a terminal is allowed to be any function of its observations, and of all previous communication. Let  $\mathbf{F}$  denote collectively all the public communication.

Given  $\varepsilon > 0$ , the rv  $K_S$  represents an  $\varepsilon$ -secret key ( $\varepsilon$ -SK) for the terminals in  $\mathcal{D}$ , achieved with communication  $\mathbf{F}$ , if there exist rvs  $K_i = K_i(\mathbf{X}_i, \mathbf{F})$ ,  $i \in \mathcal{D}$ , with  $K_i$  and  $K_S$  taking values in the same finite set  $\mathcal{K}_S$ , such that  $K_S$  satisfies

- the common randomness condition

$$\Pr\{K_i = K_S, i \in \mathcal{D}\} \geq 1 - \varepsilon;$$

- the secrecy condition

$$\frac{1}{n}I(K_S \wedge \mathbf{F}) \leq \varepsilon;$$

and

- the uniformity condition

$$\frac{1}{n}H(K_S) \geq \frac{1}{n}\log |\mathcal{K}_S| - \varepsilon.$$

Let  $A \subset \mathcal{D}$  be an arbitrary subset of the terminals. The rv  $K_P(A)$  represents an  $\varepsilon$ -private key ( $\varepsilon$ -PK) for the terminals in  $A$ , private from the terminals in  $A^c = \mathcal{D} \setminus A$ , achieved with communication  $\mathbf{F}$ , if there exist rvs  $K_i = K_i(\mathbf{X}_i, \mathbf{F})$ ,  $i \in A$ , with  $K_i$  and  $K_P(A)$  taking values in the same finite set  $\mathcal{K}_P(A)$ , such that  $K_P(A)$  satisfies

- the common randomness condition

$$\Pr\{K_i = K_P(A), i \in A\} \geq 1 - \varepsilon;$$

- the secrecy condition

$$\frac{1}{n}I(K_P(A) \wedge \{\mathbf{X}_i, i \in A^c\}, \mathbf{F}) \leq \varepsilon;$$

and

- the uniformity condition

$$\frac{1}{n}H(K_P(A)) \geq \frac{1}{n}\log |\mathcal{K}_P(A)| - \varepsilon.$$

**Definition 1** [8]: A nonnegative number  $R$  is called an *achievable SK rate* if  $\varepsilon_n$ -SKs  $K_S^{(n)}$  are achievable with suitable communication (with the number of rounds possibly depending on  $n$ ), such that  $\varepsilon_n \rightarrow 0$  and  $\frac{1}{n}H(K_S^{(n)}) \rightarrow R$ . The largest achievable SK rate is called the *SK capacity*, denoted by  $C_S$ . The PK capacity for the terminals in  $A$ , denoted by  $C_P(A)$ , is similarly defined. An achievable SK rate (resp. PK rate) will be called *strongly achievable* if  $\varepsilon_n$  above can be taken to vanish exponentially in  $n$ . The corresponding capacities are termed *strong capacities*.

Single-letter characterizations have been obtained for  $C_S$  in the case of  $d = 2$  terminals in [2], [23] and for  $d \geq 2$  terminals in [8], given by (1); and for  $C_P(A)$  in the case of  $d = 3$  terminals in [2] and for  $d \geq 3$  terminals in [8], given by (4). The proofs of the achievability parts exploit the close connection between secrecy generation and SW data compression. Loosely speaking, common randomness

sans any secrecy restrictions is first generated through SW-compressed interterminal communication, whereby all the  $d$  terminals acquire a (common) rv with probability  $\cong 1$ . In the next step, secrecy is then extracted by means of a suitable *identical* operation performed at each terminal on the acquired common randomness. When the common randomness initially acquired by the  $d$  terminals is maximal, the corresponding SK has the best rate  $C_S$  given by (1).

In this work, we consider four simple models for which we illustrate the constructions of appropriate *strong* SKs or PKs.

### B. Linear Codes for the Binary Symmetric Channel

The SW codes of interest will rely on the following classic result concerning the existence of “good” linear channel codes for a BSC. A BSC with crossover probability  $p$ ,  $0 < p < \frac{1}{2}$ , will be denoted by BSC( $p$ ). Let  $h(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$  denote the binary entropy function.

**Lemma 1** [9]: For every  $\varepsilon > 0$ ,  $0 < p < \frac{1}{2}$ , and for all  $n$  sufficiently large, there exists a binary linear  $(n, n - m)$  code for a BSC( $p$ ), with  $m < n[h(p) + \varepsilon]$ , such that the average error probability of maximum likelihood decoding is less than  $2^{-n\eta}$ , for some  $\eta > 0$ . ■

### C. Types and Typical Sequences

The following standard facts regarding “types” and “typical sequences” and their pertinent properties (cf. e.g., [6]) are compiled here in brief for ready reference.

Given finite sets  $\mathcal{X}$ ,  $\mathcal{Y}$ , the *type* of a sequence  $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$ ,  $\mathcal{X}$  a finite set, is the probability mass function (pmf)  $P_{\mathbf{x}}$  on  $\mathcal{X}$  given by

$$P_{\mathbf{x}}(a) = \frac{1}{n}|\{i : x_i = a\}|, \quad a \in \mathcal{X},$$

and the *joint type* of a pair of sequences  $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$  is the joint pmf  $P_{\mathbf{xy}}$  on  $\mathcal{X} \times \mathcal{Y}$  given by

$$P_{\mathbf{xy}}(a, b) = \frac{1}{n}|\{i : x_i = a, y_i = b\}|, \quad a \in \mathcal{X}, b \in \mathcal{Y}.$$

The numbers of different types of sequences in  $\mathcal{X}^n$  (resp.  $\mathcal{X}^n \times \mathcal{Y}^n$ ) do not exceed  $(n + 1)^{|\mathcal{X}|}$  (resp.  $(n + 1)^{|\mathcal{X}| + |\mathcal{Y}|}$ ).

Given rvs  $X, Y$  (taking values in  $\mathcal{X}, \mathcal{Y}$ , respectively), with joint pmf  $P_{XY}$  on  $\mathcal{X} \times \mathcal{Y}$ , the set of sequences in  $\mathcal{X}^n$  which are *X-typical with constant  $\xi$* , denoted by  $T_{X, \xi}^n$ , is defined as

$$T_{X, \xi}^n \triangleq \left\{ \mathbf{x} \in \mathcal{X}^n : 2^{-n[H(X) + \xi]} \leq P_X^n(\mathbf{x}) \leq 2^{-n[H(X) - \xi]} \right\},$$

where  $P_X^n(\mathbf{x}) \triangleq \Pr\{\mathbf{X} = \mathbf{x}\}$ ,  $\mathbf{x} \in \mathcal{X}^n$ ; and the set of pairs of sequences in  $\mathcal{X}^n \times \mathcal{Y}^n$  which are *XY-typical with constant  $\xi$* , denoted by  $T_{XY, \xi}^n$ , is defined as

$$T_{XY, \xi}^n \triangleq \left\{ (\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n : \mathbf{x} \in T_{X, \xi}^n, \mathbf{y} \in T_{Y, \xi}^n, \right. \\ \left. 2^{-n[H(X, Y) + \xi]} \leq P_{XY}^n(\mathbf{x}, \mathbf{y}) \leq 2^{-n[H(X, Y) - \xi]} \right\},$$

where  $P_{XY}^n(\mathbf{x}, \mathbf{y}) \triangleq \Pr\{\mathbf{X} = \mathbf{x}, \mathbf{Y} = \mathbf{y}\}$ ,  $\mathbf{x} \in \mathcal{X}^n, \mathbf{y} \in \mathcal{Y}^n$ . It readily follows that for every  $(\mathbf{x}, \mathbf{y}) \in T_{XY, \xi}^n$ ,

$$2^{-n[H(X|Y) + 2\xi]} \leq P_{X|Y}^n(\mathbf{x}|\mathbf{y}) \leq 2^{-n[H(X|Y) - 2\xi]},$$

where  $P_{X|Y}^n(\mathbf{x}|\mathbf{y}) \triangleq \Pr\{\mathbf{X} = \mathbf{x} | \mathbf{Y} = \mathbf{y}\}$ ,  $\mathbf{x} \in \mathcal{X}^n$ ,  $\mathbf{y} \in \mathcal{Y}^n$ .

For every  $\mathbf{y} \in \mathcal{Y}^n$ , the set of sequences in  $\mathcal{X}^n$  which are  $X|Y$ -typical with respect to  $\mathbf{y}$  with constant  $\xi$ , denoted by  $T_{X|Y,\xi}^n(\mathbf{y})$ , is defined as

$$T_{X|Y,\xi}^n(\mathbf{y}) \triangleq \{\mathbf{x} \in \mathcal{X}^n : (\mathbf{x}, \mathbf{y}) \in T_{XY,\xi}^n\},$$

with  $T_{X|Y,\xi}^n(\mathbf{y}) = \emptyset$  if  $\mathbf{y} \notin T_{Y,\xi}^n$ . The following is an independent and explicit statement of the well-known fact that the probability of a nontypical set decays to 0 exponentially rapidly in  $n$  (cf. e.g., [42, Theorem 6.3]).

**Proposition 1:** Given a joint pmf  $P_{XY}$  on  $\mathcal{X} \times \mathcal{Y}$  with  $P_{XY}(x, y) > 0$ ,  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ , for every  $\xi > 0$ ,

$$\sum_{\mathbf{x} \in T_{X,\xi}^n} P_X^n(\mathbf{x}) \geq 1 - (n+1)|\mathcal{X}| \cdot 2^{-n \frac{\xi^2}{2 \ln 2 \left[ \sum_{a \in \mathcal{X}} \log \frac{1}{P_X(a)} \right]^2}}, \quad (7)$$

and

$$\begin{aligned} & \sum_{(\mathbf{x}, \mathbf{y}) \in T_{XY,\xi}^n} P_{XY}^n(\mathbf{x}, \mathbf{y}) \\ & \geq 1 - (n+1)|\mathcal{X}||\mathcal{Y}| \cdot 2^{-n \frac{\xi^2}{2 \ln 2 \left[ \sum_{(a,b) \in \mathcal{X} \times \mathcal{Y}} \log \frac{1}{P_{XY}(a,b)} \right]^2}}, \quad (8) \end{aligned}$$

for all  $n \geq 1$ .

**Proof:** See Appendix A. ■

### III. MAIN RESULTS

We now present our main results on SK generation for three specific models, and PK generation for a fourth model. The proofs of the accompanying Theorems 1 - 4 are provided in Section IV.

*Model 1: Let the terminals 1 and 2 observe, respectively,  $n$  i.i.d. repetitions of the  $\{0, 1\}$ -valued rvs  $X_1$  and  $X_2$  with joint pmf*

$$\begin{aligned} P_{X_1 X_2}(x_1, x_2) &= \frac{1}{2}(1-p)\delta_{x_1 x_2} + \frac{1}{2}p(1-\delta_{x_1 x_2}), \\ 0 &< p < \frac{1}{2}, \quad (9) \end{aligned}$$

with  $\delta$  being the Kronecker delta function. These terminals wish to generate a strong SK of maximum rate.

The (strong) SK capacity for this model [2], [8], [23], given by (1), is

$$C_S = I(X_1 \wedge X_2) = 1 - h(p).$$

We show a simple scheme for the terminals to generate a SK with rate close to  $1 - h(p)$ , which relies on Wyner's well-known method for SW data compression [37]. The SW problem of interest entails terminal 2 reconstructing the observed sequence  $\mathbf{x}_1$  at terminal 1 from the SW codeword for  $\mathbf{x}_1$  and its own observed sequence  $\mathbf{x}_2$ .

Observe that under the given joint pmf (9),  $\mathbf{X}_2$  can be considered as an input to a virtual BSC( $p$ ), with corresponding output  $\mathbf{X}_1$ , i.e., we can write

$$\mathbf{X}_1 = \mathbf{X}_2 \oplus \mathbf{V}, \quad (10)$$

where  $\mathbf{V} = (V_1, \dots, V_n)$  is an i.i.d. sequence of  $\{0, 1\}$ -valued rvs, independent of  $\mathbf{X}_2$ , and with  $\Pr\{V_i = 1\} = p$ ,  $1 \leq i \leq n$ .

(i) *SW data compression* [37]: Let  $\mathcal{C}$  be a linear  $(n, n-m)$  code as in Lemma 1 with parity check matrix  $\mathbf{P}$ . Both terminals know  $\mathcal{C}$  (and  $\mathbf{P}$ ). Terminal 1 communicates the syndrome  $\mathbf{P}\mathbf{x}_1^t$  to terminal 2. The maximum likelihood estimate of  $\mathbf{x}_1$  at terminal 2 is:

$$\hat{\mathbf{x}}_2(1) = \mathbf{x}_2 \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t),$$

where  $f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t)$  is the most likely sequence  $\mathbf{v} \in \{0, 1\}^n$  (under the pmf of  $\mathbf{V}$  as above) with syndrome  $\mathbf{P}\mathbf{v}^t = \mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t$ , with  $\oplus$  denoting addition modulo 2 and  $t$  denoting transposition. Note that in a standard array corresponding to the code  $\mathcal{C}$  above,  $f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t)$  is simply the coset leader of the coset with syndrome  $\mathbf{P}\mathbf{x}_1^t \oplus \mathbf{P}\mathbf{x}_2^t$ . Also,  $\mathbf{x}_1$  and  $\hat{\mathbf{x}}_2(1)$  lie in the same coset.

The probability of decoding error at terminal 2 is given by

$$\Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} = \Pr\{\mathbf{X}_2 \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{X}_1^t \oplus \mathbf{P}\mathbf{X}_2^t) \neq \mathbf{X}_1\},$$

and it readily follows from (10) that

$$\Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} = \Pr\{f_{\mathbf{P}}(\mathbf{P}\mathbf{V}^t) \neq \mathbf{V}\}.$$

By Lemma 1,  $\Pr\{f_{\mathbf{P}}(\mathbf{P}\mathbf{V}^t) \neq \mathbf{V}\} < 2^{-n\eta}$  for some  $\eta > 0$  and for all  $n$  sufficiently large, so that

$$\Pr\{\hat{\mathbf{X}}_2(1) = \mathbf{X}_1\} \geq 1 - 2^{-n\eta}.$$

(ii) *SK construction:* Consider a (common) standard array for  $\mathcal{C}$  known to both terminals. Denote by  $\mathbf{a}_{i,j}$  the element of the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column in the standard array,  $1 \leq i \leq 2^m$ ,  $1 \leq j \leq 2^{n-m}$ .

Terminal 1 sets  $K_1 = j_1$  if  $\mathbf{X}_1$  equals  $\mathbf{a}_{i,j_1}$  in its coset  $i$  in the standard array. Terminal 2 sets  $K_2 = j_2$  if  $\hat{\mathbf{X}}_2(1)$  equals  $\mathbf{a}_{i,j_2}$  in the coset  $i$  of the same standard array.

The following theorem asserts that  $K_1$  constitutes a strong SK with rate approaching SK capacity.

**Theorem 1:** Let  $\varepsilon > 0$  be given. Then for some  $\eta > 0$  and for all  $n$  sufficiently large, the pair of rvs  $(K_1, K_2)$  generated above, with (common) range  $\mathcal{K}_1$  (say), satisfy

$$\Pr\{K_1 = K_2\} \geq 1 - 2^{-n\eta}, \quad (11)$$

$$I(K_1 \wedge \mathbf{F}) = 0, \quad (12)$$

$$H(K_1) = \log |\mathcal{K}_1|, \quad (13)$$

and

$$\frac{1}{n}H(K_1) > 1 - h(p) - \varepsilon. \quad (14)$$

*Remark:* The probability of  $K_1$  differing from  $K_2$  equals exactly the average error probability of maximum likelihood decoding when  $\mathcal{C}$  is used on a BSC( $p$ ). Furthermore, the gap between the rate of the generated SK and SK capacity equals the gap between the rate of  $\mathcal{C}$  and channel capacity.

*Model 2: Let the terminals 1 and 2 observe, respectively,  $n$  i.i.d. repetitions of the  $\{0, 1\}$ -valued rvs with joint pmf*

$$\begin{aligned} P_{X_1 X_2}(0, 0) &= (1-p)(1-q), \\ P_{X_1 X_2}(0, 1) &= pq, \\ P_{X_1 X_2}(1, 0) &= p(1-q), \\ P_{X_1 X_2}(1, 1) &= q(1-p), \end{aligned} \quad (15)$$

with  $0 < p < \frac{1}{2}$  and  $0 < q < 1$ . These terminals wish to generate a strong SK of maximum rate.

Note that Model 1 is a special case of Model 2 for  $q = \frac{1}{2}$ . We show below a scheme for the terminals to generate a SK with rate close to the (strong) SK capacity for this model [2], [8], [23], which is given by (1) as

$$C_S = I(X_1 \wedge X_2) = h(p + q - 2pq) - h(p).$$

(i) *SW data compression*: This step is identical to step (i) for Model 1. Note that under the given joint pmf (15),  $\mathbf{X}_1$  and  $\mathbf{X}_2$  can be written as in (10). It follows in the same manner as for Model 1 that for some  $\eta > 0$  and for all  $n$  sufficiently large,

$$\Pr\{\hat{\mathbf{X}}_2(1) = \mathbf{X}_1\} \geq 1 - 2^{-n\eta}.$$

(ii) *SK construction*: Both terminals know the linear  $(n, n-m)$  code  $\mathcal{C}$  as in Lemma 1, and a (common) standard array for  $\mathcal{C}$ . Let  $\{\mathbf{e}_i : 1 \leq i \leq 2^m\}$  denote the set of coset leaders for all the cosets of  $\mathcal{C}$ .

Denote by  $A_i$  the set of sequences from  $T_{X_1, \xi}^n$  in the coset of  $\mathcal{C}$  with coset leader  $\mathbf{e}_i$ ,  $1 \leq i \leq 2^m$ . If the number of sequences of the same type in  $A_i$  is more than  $2^{n[I(X_1 \wedge X_2) - \varepsilon']}$ , where  $\varepsilon' > \xi + \varepsilon$  with  $\varepsilon$  satisfying  $m < n[h(p) + \varepsilon]$  in Lemma 1, then collect arbitrarily  $2^{n[I(X_1 \wedge X_2) - \varepsilon']}$  such sequences to compose a subset, which we term a *regular subset* (as it consists of sequences of the same type). Continue this procedure until the number of sequences of every type in  $A_i$  is less than  $2^{n[I(X_1 \wedge X_2) - \varepsilon']}$ . Let  $N_i$  denote the number of distinct regular subsets of  $A_i$ .

Enumerate (in any way) the sequences in each regular subset. Let  $\mathbf{b}_{i,j,k}$ , where  $1 \leq i \leq 2^m$ ,  $1 \leq j \leq N_i$ ,  $1 \leq k \leq 2^{n[I(X_1 \wedge X_2) - \varepsilon']}$ , denote the  $k^{\text{th}}$  sequence of the  $j^{\text{th}}$  regular subset in the  $i^{\text{th}}$  coset (with coset leader  $\mathbf{e}_i$ ).

Terminal 1 sets  $K_1 = k_1$  if  $\mathbf{X}_1$  equals  $\mathbf{b}_{i,j_1,k_1}$ ; else,  $K_1$  is set to be uniformly distributed on  $\{1, \dots, 2^{n[I(X_1 \wedge X_2) - \varepsilon']}\}$ , independent of  $(\mathbf{X}_1, \mathbf{X}_2)$ . Terminal 2 sets  $K_2 = k_2$  if  $\hat{\mathbf{X}}_2(1)$  equals  $\mathbf{b}_{i,j_2,k_2}$ ; else,  $K_2$  is set to be uniformly distributed on  $\{1, \dots, 2^{n[I(X_1 \wedge X_2) - \varepsilon']}\}$ , independent of  $(\mathbf{X}_1, \mathbf{X}_2, K_1)$ .

The following theorem says that  $K_1$  constitutes a strong SK with rate approaching SK capacity.

**Theorem 2:** Let  $\varepsilon > 0$  be given. Then for some  $\eta' = \eta'(\eta, \xi, \varepsilon, \varepsilon') > 0$  and for all  $n$  sufficiently large, the pair of rvs  $(K_1, K_2)$  generated above, with range  $\mathcal{K}_1$  (say), satisfy

$$\Pr\{K_1 = K_2\} \geq 1 - 2^{-n\eta'}, \quad (16)$$

$$I(K_1 \wedge \mathbf{F}) = 0, \quad (17)$$

$$H(K_1) = \log |\mathcal{K}_1|, \quad (18)$$

and

$$\frac{1}{n}H(K_1) = h(p + q - 2pq) - h(p) - \varepsilon'. \quad (19)$$

The next model is an instance of a *Markov chain on a tree* (cf. [13], [8]). Consider a tree  $\mathcal{T}$  with vertex set  $V(\mathcal{T}) = \{1, \dots, d\}$  and edge set  $E(\mathcal{T})$ . For  $(i, j) \in E(\mathcal{T})$ , let  $B(i \leftarrow j)$  denote the set of all vertices connected with  $j$  by a path

containing the edge  $(i, j)$ . The rvs  $X_1, \dots, X_d$  form a *Markov chain on the tree*  $\mathcal{T}$  if for each  $(i, j) \in E(\mathcal{T})$ , the conditional pmf of  $X_j$  given  $\{X_l, l \in B(i \leftarrow j)\}$  depends only on  $X_i$  (i.e., is conditionally independent of  $\{X_l, l \in B(i \leftarrow j)\} \setminus \{X_i\}$ , conditioned on  $X_i$ ). Note that when  $\mathcal{T}$  is a chain, this concept reduces to that of a standard Markov chain.

**Model 3:** Let the terminals  $1, \dots, d$  observe, respectively,  $n$  i.i.d. repetitions of  $\{0, 1\}$ -valued rvs  $X_1, \dots, X_d$  that form a Markov chain on the tree  $\mathcal{T}$ , with joint pmf  $P_{X_1 \dots X_d}$  specified as: for  $(i, j) \in E(\mathcal{T})$ ,

$$P_{X_i X_j}(x_i, x_j) = \frac{1}{2}(1 - p_{(i,j)})\delta_{x_i x_j} + \frac{1}{2}p_{(i,j)}(1 - \delta_{x_i x_j}),$$

$$0 < p_{(i,j)} < \frac{1}{2},$$

for  $x_i, x_j \in \{0, 1\}$ . These  $d$  terminals wish to generate a strong SK of maximum rate.

Note that Model 1 is a special case of Model 3 for  $d = 2$ . Without any loss of generality, let

$$p_{\max} = p_{(i^*, j^*)} = \max_{(i,j) \in E(\mathcal{T})} p_{(i,j)}.$$

Then, the (strong) SK capacity for this model [8] is given by (1) as

$$C_S = I(X_{i^*} \wedge X_{j^*}) = 1 - h(p_{\max}).$$

We show how to extract a SK with rate close to  $1 - h(p_{\max})$  by using an extension of the SW data compression scheme of Model 1 for reconstructing  $\mathbf{x}_{i^*}$  at all the terminals.

(i) *SW data compression*: Let  $\mathcal{C}$  be the linear  $(n, n-m)$  code as in Lemma 1 for a BSC( $p_{\max}$ ), and with parity check matrix  $\mathbf{P}$ . Each terminal  $i$  communicates the syndrome  $\mathbf{P}\mathbf{x}_i^t$ ,  $1 \leq i \leq d$ .

Let  $\hat{\mathbf{x}}_i(j)$  denote the corresponding maximum likelihood estimate of  $\mathbf{x}_j$  at terminal  $i$ ,  $1 \leq i \neq j \leq d$ . For a terminal  $i \neq i^*$ , denote by  $(i_0, i_1, \dots, i_r)$  the (only) path in the tree  $\mathcal{T}$  from  $i$  to  $i^*$ , where  $i_0 = i$  and  $i_r = i^*$ ; this terminal  $i$ , with the knowledge of  $(\mathbf{x}_i, \mathbf{P}\mathbf{x}_i^t, \dots, \mathbf{P}\mathbf{x}_{i_{r-1}}^t, \mathbf{P}\mathbf{x}_{i^*}^t)$ , forms its estimate  $\hat{\mathbf{x}}_i(i^*)$  of  $\mathbf{x}_{i^*}$  through the following successive maximum likelihood estimates of  $\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_{r-1}}$ :

$$\begin{aligned} \hat{\mathbf{x}}_i(i_1) &= \mathbf{x}_i \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_i^t \oplus \mathbf{P}\mathbf{x}_{i_1}^t), \\ \hat{\mathbf{x}}_i(i_2) &= \hat{\mathbf{x}}_i(i_1) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{i_1}^t \oplus \mathbf{P}\mathbf{x}_{i_2}^t), \\ &\vdots \\ \hat{\mathbf{x}}_i(i_{r-1}) &= \hat{\mathbf{x}}_i(i_{r-2}) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{i_{r-2}}^t \oplus \mathbf{P}\mathbf{x}_{i_{r-1}}^t), \end{aligned}$$

and finally,

$$\hat{\mathbf{x}}_i(i^*) = \hat{\mathbf{x}}_i(i_{r-1}) \oplus f_{\mathbf{P}}(\mathbf{P}\mathbf{x}_{i_{r-1}}^t \oplus \mathbf{P}\mathbf{x}_{i^*}^t). \quad (20)$$

**Proposition 2:** By the successive maximum likelihood estimation above, the estimate  $\hat{\mathbf{X}}_i(i^*)$  at terminal  $i \neq i^*$ , satisfies

$$\Pr\{\hat{\mathbf{X}}_i(i^*) = \mathbf{X}_{i^*}\} \geq 1 - d \cdot 2^{-n\eta}, \quad (21)$$

for some  $\eta > 0$  and for all  $n$  sufficiently large.

**Proof:** See Appendix B. ■

It follows directly from (21) that for some  $\eta' = \eta'(\eta, m) > 0$  and for all  $n$  sufficiently large,

$$\Pr\{\hat{\mathbf{X}}_i(i^*) = \mathbf{X}_{i^*}, 1 \leq i \neq i^* \leq d\} \geq 1 - 2^{-n\eta'}.$$

(ii) *SK construction*: Consider a (common) standard array for  $\mathcal{C}$  known to all the terminals. Denote by  $\mathbf{a}_{l,k}$  the element of the  $l^{\text{th}}$  row and the  $k^{\text{th}}$  column in the standard array,  $1 \leq l \leq 2^m$ ,  $1 \leq k \leq 2^{n-m}$ . Terminal  $i^*$  sets  $K_{i^*} = k_{i^*}$  if  $\mathbf{X}_{i^*}$  equals  $\mathbf{a}_{l,k_{i^*}}$  in the standard array. Terminal  $i$ ,  $1 \leq i \neq i^* \leq d$ , sets  $K_i = k_i$  if  $\hat{\mathbf{X}}_i(i^*)$  equals  $\mathbf{a}_{l,k_i}$  in the same standard array.

The following theorem states that  $K_{i^*}$  constitutes a strong SK with rate approaching SK capacity.

**Theorem 3**: Let  $\varepsilon > 0$  be given. Then for some  $\eta' = \eta'(\eta, d) > 0$  and for all  $n$  sufficiently large, the rvs  $K_1, \dots, K_d$  generated above, with range  $\mathcal{K}_{i^*}$  (say), satisfy

$$\Pr\{K_1 = \dots = K_d\} > 1 - 2^{-n\eta'}, \quad (22)$$

$$I(K_{i^*} \wedge \mathbf{F}) = 0, \quad (23)$$

$$H(K_{i^*}) = \log |\mathcal{K}_{i^*}|, \quad (24)$$

and

$$\frac{1}{n}H(K_{i^*}) > 1 - h(p_{\max}) - \varepsilon. \quad (25)$$

*Model 4*: Let the terminals 1, 2 and 3 observe, respectively,  $n$  i.i.d. repetitions of the  $\{0, 1\}$ -valued rvs  $X_1, X_2, X_3$ , with joint pmf  $P_{X_1 X_2 X_3}$  given by:

$$\begin{aligned} P_{X_1 X_2 X_3}(0, 0, 0) &= P_{X_1 X_2 X_3}(0, 1, 1) = \frac{(1-p)(1-q)}{2}, \\ P_{X_1 X_2 X_3}(0, 0, 1) &= P_{X_1 X_2 X_3}(0, 1, 0) = \frac{pq}{2}, \\ P_{X_1 X_2 X_3}(1, 0, 0) &= P_{X_1 X_2 X_3}(1, 1, 1) = \frac{p(1-q)}{2}, \\ P_{X_1 X_2 X_3}(1, 0, 1) &= P_{X_1 X_2 X_3}(1, 1, 0) = \frac{q(1-p)}{2}, \end{aligned} \quad (26)$$

with  $0 < p < \frac{1}{2}$  and  $0 < q < 1$ . Terminals 1 and 2 wish to generate a strong PK of maximum rate, which is concealed from the helper terminal 3.

Note that under the joint pmf of  $X_1, X_2, X_3$  above, we can write

$$\mathbf{X}_1 = \mathbf{X}_2 \oplus \mathbf{X}_3 \oplus \mathbf{V}, \quad (27)$$

where  $\mathbf{V} = (V_1, \dots, V_n)$  is an i.i.d. sequence of  $\{0, 1\}$ -valued rvs, independent of  $(\mathbf{X}_2, \mathbf{X}_3)$ , with  $\Pr\{V_i = 1\} = p$ ,  $1 \leq i \leq n$ . Further,  $(X_2, X_3)$  plays the role of  $(X_1, X_2)$  in Model 1 with  $q$  in lieu of  $p$  in the latter.

We show below a scheme for terminals 1 and 2 to generate a PK with rate close to (strong) PK capacity for this model [2], [7], [8], given by (4) as

$$C_P(\{1, 2\}) = I(X_1 \wedge X_2 | X_3) = h(p + q - 2pq) - h(p).$$

The first step of this scheme entails terminal 3 simply revealing its observations  $\mathbf{x}_3$  to both terminals 1 and 2. Then, Wyner's SW data compression scheme is used for reconstructing  $\mathbf{x}_1$  at terminal 2 from the SW codeword for  $\mathbf{x}_1$  and its own knowledge of  $\mathbf{x}_2 \oplus \mathbf{x}_3$ .

(i) *SW data compression*: This step is identical to step (i) for Model 1, as seen with the help of (27). Obviously,

$$\Pr\{\hat{\mathbf{X}}_2(1) = \mathbf{X}_1\} \geq 1 - 2^{-n\eta},$$

for some  $\eta > 0$  and for all  $n$  sufficiently large.

(ii) *PK construction*: Suppose that terminals 1 and 2 know a linear  $(n, n - m)$  code  $\mathcal{C}$  as in Lemma 1, and a (common) standard array for  $\mathcal{C}$ . Let  $\{\mathbf{e}_i : 1 \leq i \leq 2^m\}$  denote the set of coset leaders for all the cosets of  $\mathcal{C}$ .

For a sequence  $\mathbf{x}_3 \in \{0, 1\}^n$ , denote by  $A_i(\mathbf{x}_3)$  the set of sequences from  $T_{X_1|X_3, \xi}^n(\mathbf{x}_3)$  in the coset of  $\mathcal{C}$  with coset leader  $\mathbf{e}_i$ ,  $1 \leq i \leq 2^m$ . If the number of sequences of the same joint type with  $\mathbf{x}_3$  in  $A_i(\mathbf{x}_3)$  is more than  $2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$ , where  $\varepsilon' > 2\xi + \varepsilon$  and  $\varepsilon$  satisfies  $m < n[h(p) + \varepsilon]$  (as in Lemma 1), then collect arbitrarily  $2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$  such sequences to compose a regular subset. Continue this procedure until the number of sequences of every joint type with  $\mathbf{x}_3$  in  $A_i(\mathbf{x}_3)$  is less than  $2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$ . Let  $N_i(\mathbf{x}_3)$  denote the number of distinct regular subsets of  $A_i(\mathbf{x}_3)$ .

For a given sequence  $\mathbf{x}_3$ , enumerate (in any way) the sequences in each regular subset. Let  $\mathbf{b}_{i,j,k}(\mathbf{x}_3)$ , where  $1 \leq i \leq 2^m$ ,  $1 \leq j \leq N_i(\mathbf{x}_3)$ ,  $1 \leq k \leq 2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}$ , denote the  $k^{\text{th}}$  sequence of the  $j^{\text{th}}$  regular subset in the  $i^{\text{th}}$  coset.

Terminal 1 sets  $K_1 = k_1$  if  $\mathbf{X}_1$  equals  $\mathbf{b}_{i,j_1,k_1}(\mathbf{x}_3)$ ; else,  $K_1$  is set to be uniformly distributed on  $\{1, \dots, 2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}\}$ , independent of  $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ . Terminal 2 sets  $K_2 = k_2$  if  $\hat{\mathbf{X}}_2(1)$  equals  $\mathbf{b}_{i,j_2,k_2}(\mathbf{x}_3)$ ; else,  $K_2$  is set to be uniformly distributed on  $\{1, \dots, 2^{n[I(X_1 \wedge X_2 | X_3) - \varepsilon']}\}$ , independent of  $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, K_1)$ .

The following theorem establishes that  $K_1$  constitutes a strong PK with rate approaching PK capacity.

**Theorem 4**: Let  $\varepsilon > 0$  be given. Then for some  $\eta' = \eta'(\eta, \xi, \varepsilon, \varepsilon') > 0$  and for all  $n$  sufficiently large, the pair of rvs  $(K_1, K_2)$  generated above, with range  $\mathcal{K}_1$  (say), satisfy

$$\Pr\{K_1 \neq K_2\} < 2^{-n\eta'}, \quad (28)$$

$$I(K_1 \wedge \mathbf{X}_3, \mathbf{F}) = 0, \quad (29)$$

$$H(K_1) = \log |\mathcal{K}_1|, \quad (30)$$

and

$$\frac{1}{n}H(K_1) = I(X_1 \wedge X_2 | X_3) - \varepsilon'. \quad (31)$$

*Remark*: The PK construction scheme above applies for any joint pmf  $P_{X_1 X_2 X_3}$  satisfying (27), and is not restricted to the given joint pmf in (26).

#### IV. PROOFS OF THEOREMS 1–4

**Proof of Theorem 1**: It follows from the SK construction scheme for Model 1 that

$$\Pr\{K_1 \neq K_2\} = \Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} < 2^{-n\eta},$$

which is (11). Since  $X_1$  is uniformly distributed on  $\{0, 1\}$ , we have for  $1 \leq i \leq 2^m$ ,  $1 \leq j \leq 2^{n-m}$ , that

$$\Pr\{\mathbf{X}_1 = \mathbf{a}_{i,j}\} = 2^{-n}.$$

Hence,

$$\begin{aligned}\Pr\{K_1 = j\} &= \sum_{i=1}^{2^m} \Pr\{\mathbf{X}_1 = \mathbf{a}_{i,j}\} \\ &= 2^{-(n-m)}, \quad 1 \leq j \leq 2^{n-m},\end{aligned}$$

i.e.,  $K_1$  is uniformly distributed on  $\mathcal{K}_1 = \{1, \dots, 2^{n-m}\}$ , and so

$$H(K_1) = \log 2^{n-m} = n - m = \log |\mathcal{K}_1|,$$

which is (13). Therefore, (14) holds since  $m < n[h(p) + \varepsilon]$ .

It remains to show that  $K_1$  satisfies (12) with  $\mathbf{F} = \mathbf{P}\mathbf{X}_1^t$ . Let  $\{\mathbf{e}_i, 1 \leq i \leq 2^m\}$  be the set of coset leaders for the cosets of  $\mathcal{C}$ . For  $1 \leq i \leq 2^m, 1 \leq j \leq 2^{n-m}$ ,

$$\begin{aligned}\Pr\{K_1 = j | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t\} &= \frac{\Pr\{K_1 = j, \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t\}}{\Pr\{\mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t\}} \\ &= \frac{\Pr\{\mathbf{X}_1 = \mathbf{a}_{i,j}\}}{\sum_{j'=1}^{2^{n-m}} \Pr\{\mathbf{X}_1 = \mathbf{a}_{i,j'}\}} \\ &= 2^{-(n-m)} \\ &= \Pr\{K_1 = j\},\end{aligned}$$

i.e.,  $K_1$  is independent of  $\mathbf{F}$ , and so  $I(K_1 \wedge \mathbf{F}) = 0$ , establishing (12). ■

**Proof of Theorem 2:** Let  $\mathcal{F}$  denote the union of all regular subsets in  $\bigcup_{i=1}^{2^m} A_i$ . Clearly  $\mathcal{F} \subseteq T_{X_1, \xi}^n$ , so that

$$\begin{aligned}\Pr\{\mathbf{X}_1 \in \mathcal{F}\} &= \Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n, \mathbf{X}_1 \in \mathcal{F}\} \\ &= \Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n\} - \Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n \setminus \mathcal{F}\}.\end{aligned}\quad (32)$$

By Proposition 1,  $\Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n\}$  goes to 1 exponentially rapidly in  $n$ . We show below that  $\Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n \setminus \mathcal{F}\}$  decays to 0 exponentially rapidly in  $n$ .

Since the number of different types of sequences in  $\{0, 1\}^n$  does not exceed  $(n+1)^2$ , we have that

$$\begin{aligned}|\{\mathbf{x}_1 : \mathbf{x}_1 \in T_{X_1, \xi}^n \setminus \mathcal{F}\}| &\leq 2^m \cdot (n+1)^2 \cdot 2^{n[I(X_1 \wedge X_2) - \varepsilon']} \\ &< (n+1)^2 \cdot 2^{n[H(X_1) + \varepsilon - \varepsilon']},\end{aligned}$$

where the previous inequality is from  $m < n[h(p) + \varepsilon] = n[H(X_1|X_2) + \varepsilon]$ .

Since  $P_{X_1}^n(\mathbf{x}_1) \leq 2^{-n[H(X_1) - \xi]}$ ,  $\mathbf{x}_1 \in T_{X_1, \xi}^n$ , we get

$$\Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n \setminus \mathcal{F}\} < (n+1)^2 \cdot 2^{-n(\varepsilon' - \xi - \varepsilon)}.$$

Choosing  $\varepsilon' > \xi + \varepsilon$ ,  $\Pr\{\mathbf{X}_1 \in T_{X_1, \xi}^n \setminus \mathcal{F}\}$  goes to 0 exponentially rapidly. Therefore, it follows from (32) that  $\Pr\{\mathbf{X}_1 \in \mathcal{F}\}$  goes to 1 exponentially rapidly in  $n$ , with exponent depending on  $(\xi, \varepsilon, \varepsilon')$ .

By the SK construction scheme for Model 2,

$$\begin{aligned}\Pr\{K_1 \neq K_2\} &= \Pr\{K_1 \neq K_2, \mathbf{X}_1 \in \mathcal{F}\} + \Pr\{K_1 \neq K_2, \mathbf{X}_1 \notin \mathcal{F}\} \\ &\leq \Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1, \mathbf{X}_1 \in \mathcal{F}\} + \Pr\{\mathbf{X}_1 \notin \mathcal{F}\} \\ &\leq \Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} + \Pr\{\mathbf{X}_1 \notin \mathcal{F}\}.\end{aligned}$$

Since  $\Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} < 2^{-n\eta}$ , by the observation in the previous paragraph, we have

$$\Pr\{K_1 \neq K_2\} < 2^{-n\eta'}$$

for some  $\eta' = \eta'(\eta, \xi, \varepsilon, \varepsilon') > 0$  and for all  $n$  sufficiently large, which is (16).

Next, we shall show that  $K_1$  satisfies (18). For  $1 \leq k \leq 2^{n[I(X_1 \wedge X_2) - \varepsilon']}$ , it is clear by choice that

$$\Pr\{K_1 = k | \mathbf{X}_1 \notin \mathcal{F}\} = 2^{-n[I(X_1 \wedge X_2) - \varepsilon']}, \quad (33)$$

and that

$$\begin{aligned}\Pr\{K_1 = k | \mathbf{X}_1 \in \mathcal{F}\} &= \frac{\Pr\{K_1 = k, \mathbf{X}_1 \in \mathcal{F}\}}{\Pr\{\mathbf{X}_1 \in \mathcal{F}\}} \\ &= \frac{\sum_{i=1}^{2^m} \sum_{j=1}^{N_i} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}\}}{\sum_{i=1}^{2^m} \sum_{j=1}^{N_i} 2^{n[I(X_1 \wedge X_2) - \varepsilon']} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}\}} \\ &= 2^{-n[I(X_1 \wedge X_2) - \varepsilon']},\end{aligned}\quad (34)$$

where (34) is due to every regular subset consisting of sequences of the same type. From (33) and (35),

$$\Pr\{K_1 = k\} = 2^{-n[I(X_1 \wedge X_2) - \varepsilon']}, \quad (36)$$

i.e.,  $K_1$  is uniformly distributed on  $\mathcal{K}_1 = \{1, \dots, 2^{n[I(X_1 \wedge X_2) - \varepsilon']}\}$ , with

$$\frac{1}{n} H(K_1) = I(X_1 \wedge X_2) - \varepsilon',$$

which is (19).

It remains to show that  $K_1$  satisfies (17) with  $\mathbf{F} = \mathbf{P}\mathbf{X}_1^t$ . For  $1 \leq i \leq 2^m, 1 \leq k \leq 2^{n[I(X_1 \wedge X_2) - \varepsilon']}$ , we have

$$\Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \notin \mathcal{F}\} = 2^{-n[I(X_1 \wedge X_2) - \varepsilon']}$$

by choice, and

$$\begin{aligned}\Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}\} &= \frac{\Pr\{K_1 = k, \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}\}}{\Pr\{\mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}\}} \\ &= \frac{\sum_{j=1}^{N_i} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}\}}{\sum_{j=1}^{N_i} 2^{n[I(X_1 \wedge X_2) - \varepsilon']} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}\}} \\ &= 2^{-n[I(X_1 \wedge X_2) - \varepsilon']}.\end{aligned}$$

Hence,

$$\begin{aligned}\Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t\} &= \Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}\} \times \\ &\quad \Pr\{\mathbf{X}_1 \in \mathcal{F} | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t\} \\ &\quad + \Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \notin \mathcal{F}\} \times \\ &\quad \Pr\{\mathbf{X}_1 \notin \mathcal{F} | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t\} \\ &= 2^{-n[I(X_1 \wedge X_2) - \varepsilon']} \\ &= \Pr\{K_1 = k\},\end{aligned}$$

where the previous equality follows from (36). Thus,  $K_1$  is independent of  $\mathbf{F}$ , establishing (17). ■

**Proof of Theorem 3:** Applying the same arguments used in Theorem 1, we see that the rvs  $K_1, \dots, K_m$  satisfy (22), (24)

and (25). It then remains to show that  $K_{i^*}$  satisfies (23) with  $\mathbf{F} = (\mathbf{P}\mathbf{X}_1^t, \dots, \mathbf{P}\mathbf{X}_d^t)$ .

Under the given joint pmf  $P_{X_1 \dots X_d}$ , for each  $i \neq i^*$ , we can write

$$\mathbf{X}_i = \mathbf{X}_{i^*} \oplus \mathbf{V}_i,$$

where  $\mathbf{V}_i = (V_{i,1}, \dots, V_{i,n})$  is an i.i.d. sequence of  $\{0, 1\}$ -valued rvs. Further,  $\mathbf{V}_i$ ,  $1 \leq i \neq i^* \leq d$ , and  $\mathbf{X}_{i^*}$  are mutually independent. Then,

$$\begin{aligned} I(K_{i^*} \wedge \mathbf{F}) &= I(K_{i^*} \wedge \{\mathbf{P}\mathbf{X}_i^t, 1 \leq i \leq d\}) \\ &\leq I(K_{i^*} \wedge \mathbf{P}\mathbf{X}_{i^*}^t, \{\mathbf{P}\mathbf{V}_i^t, 1 \leq i \neq i^* \leq d\}) \\ &\leq I(K_{i^*} \wedge \mathbf{P}\mathbf{X}_{i^*}^t) \\ &\quad + I(K_{i^*}, \mathbf{P}\mathbf{X}_{i^*}^t \wedge \{\mathbf{P}\mathbf{V}_i^t, 1 \leq i \neq i^* \leq d\}). \end{aligned} \quad (37)$$

Clearly, the first term on the right hand side of (37) is zero. Since for a fixed  $\mathbf{P}$ ,  $(K_{i^*}, \mathbf{P}\mathbf{X}_{i^*}^t)$  is a function of  $\mathbf{X}_{i^*}$ ,

$$\begin{aligned} I(K_{i^*}, \mathbf{P}\mathbf{X}_{i^*}^t \wedge \{\mathbf{P}\mathbf{V}_i^t, 1 \leq i \neq i^* \leq d\}) \\ \leq I(\mathbf{X}_{i^*} \wedge \{\mathbf{V}_i, 1 \leq i \neq i^* \leq d\}) = 0, \end{aligned}$$

i.e.,  $K_{i^*}$  is independent of  $\mathbf{F}$ , establishing (23).  $\blacksquare$

**Proof of Theorem 4:** For every  $\mathbf{x}_3 \in \{0, 1\}^n$ , let  $\mathcal{F}(\mathbf{x}_3)$  denote the union of all regular subsets in  $\bigcup_{i=1}^{2^m} A_i(\mathbf{x}_3)$ . Since  $\mathcal{F}(\mathbf{x}_3) \subseteq T_{X_1|X_3, \xi}^n(\mathbf{x}_3)$ ,

$$\begin{aligned} \Pr\{\mathbf{X}_1 \in \mathcal{F}(\mathbf{X}_3)\} &= \Pr\{\mathbf{X}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{X}_3)\} \\ &\quad - \Pr\{\mathbf{X}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{X}_3) \setminus \mathcal{F}(\mathbf{X}_3)\} \end{aligned} \quad (38)$$

It follows from Proposition 1 that  $\Pr\{\mathbf{X}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{X}_3)\}$  goes to 1 exponentially rapidly in  $n$ . We show below that  $\Pr\{\mathbf{X}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{X}_3) \setminus \mathcal{F}(\mathbf{X}_3)\}$  goes to 0 exponentially rapidly in  $n$ .

Recall that the number of different joint types of pairs in  $\{0, 1\}^n \times \{0, 1\}^n$  does not exceed  $(n+1)^4$ . Thus,

$$\begin{aligned} \left| \{\mathbf{x}_1 : \mathbf{x}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{x}_3) \setminus \mathcal{F}(\mathbf{x}_3)\} \right| \\ \leq 2^m \cdot (n+1)^4 \cdot 2^{n[I(X_1 \wedge X_2|X_3) - \varepsilon']} \\ < (n+1)^4 \cdot 2^{n[H(X_1|X_3) + \varepsilon - \varepsilon']}, \end{aligned}$$

where the previous inequality is from  $m < n[h(p) + \varepsilon] = n[H(X_1|X_2, X_3) + \varepsilon]$ .

Since  $P_{X_1|X_3}^n(\mathbf{x}_1|\mathbf{x}_3) \leq 2^{-n[H(X_1|X_3) - 2\xi]}$ ,  $(\mathbf{x}_1, \mathbf{x}_3) \in T_{X_1|X_3, \xi}^n$ , we get

$$\Pr\{\mathbf{X}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{X}_3) \setminus \mathcal{F}(\mathbf{X}_3)\} < (n+1)^4 \cdot 2^{-n(\varepsilon' - 2\xi - \varepsilon)}.$$

Choosing  $\varepsilon' > 2\xi + \varepsilon$ ,  $\Pr\{\mathbf{X}_1 \in T_{X_1|X_3, \xi}^n(\mathbf{X}_3) \setminus \mathcal{F}(\mathbf{X}_3)\}$  goes to 0 exponentially rapidly. Therefore, it follows from (38) that  $\Pr\{\mathbf{X}_1 \in \mathcal{F}(\mathbf{X}_3)\}$  goes to 1 exponentially rapidly in  $n$ , with an exponent depending on  $(\xi, \varepsilon, \varepsilon')$ .

By the PK construction scheme for Model 4,

$$\begin{aligned} \Pr\{K_1 \neq K_2\} &= \Pr\{K_1 \neq K_2, \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3)\} + \Pr\{K_1 \neq K_2, \mathbf{X}_1 \notin \mathcal{F}(\mathbf{x}_3)\} \\ &\leq \Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1, \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3)\} + \Pr\{\mathbf{X}_1 \notin \mathcal{F}(\mathbf{x}_3)\} \\ &\leq \Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} + \Pr\{\mathbf{X}_1 \notin \mathcal{F}(\mathbf{X}_3)\}. \end{aligned}$$

Since  $\Pr\{\hat{\mathbf{X}}_2(1) \neq \mathbf{X}_1\} < 2^{-n\eta}$  by the observation in the previous paragraph, we have

$$\Pr\{K_1 \neq K_2\} < 2^{-n\eta'},$$

for some  $\eta' = \eta'(\eta, \xi, \varepsilon, \varepsilon') > 0$  and for all  $n$  sufficiently large, which is (28).

Next, we shall show that  $K_1$  satisfies (30). For  $\mathbf{x}_3 \in \{0, 1\}^n$  and  $1 \leq k \leq 2^{n[I(X_1 \wedge X_2|X_3) - \varepsilon']}$ , it is clear by choice that

$$\Pr\{K_1 = k | \mathbf{X}_1 \notin \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} = 2^{-n[I(X_1 \wedge X_2|X_3) - \varepsilon']},$$

and that

$$\begin{aligned} \Pr\{K_1 = k | \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} &= \frac{\Pr\{K_1 = k, \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}}{\Pr\{\mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}} \\ &= \frac{\sum_{i=1}^{2^m} \sum_{j=1}^{N_i(\mathbf{x}_3)} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}}{\sum_{i=1}^{2^m} \sum_{j=1}^{N_i(\mathbf{x}_3)} 2^{n[I(X_1 \wedge X_2|X_3) - \varepsilon']} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}} \\ &= 2^{-n[I(X_1 \wedge X_2|X_3) - \varepsilon']}, \end{aligned}$$

where the second equality is due to every regular subset consisting of sequences of the same joint type with  $\mathbf{x}_3$ . Therefore,

$$\begin{aligned} \Pr\{K_1 = k\} &= \sum_{\mathbf{x}_3 \in \{0, 1\}^n} \Pr\{K_1 = k, \mathbf{X}_3 = \mathbf{x}_3\} \\ &= \sum_{\mathbf{x}_3 \in \{0, 1\}^n} [\Pr\{\mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} \times \\ &\quad \Pr\{K_1 = k | \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} \\ &\quad + \Pr\{\mathbf{X}_1 \notin \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} \times \\ &\quad \Pr\{K_1 = k | \mathbf{X}_1 \notin \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\}] \\ &= 2^{-n[I(X_1 \wedge X_2|X_3) - \varepsilon']}, \end{aligned} \quad (39)$$

i.e.,  $K_1$  is uniformly distributed on  $\mathcal{K}_1 = \{1, \dots, 2^{n[I(X_1 \wedge X_2|X_3) - \varepsilon']}\}$ , with

$$\frac{1}{n} H(K_1) = I(X_1 \wedge X_2|X_3) - \varepsilon',$$

which is (31).

It remains to show that  $K_1$  satisfies (29) with  $(\mathbf{X}_3, \mathbf{F}) = (\mathbf{X}_3, \mathbf{P}\mathbf{X}_1^t)$ . For  $\mathbf{x}_3 \in \{0, 1\}^n$ ,  $1 \leq i \leq 2^m$  and  $1 \leq k \leq 2^{n[I(X_1 \wedge X_2|X_3) - \varepsilon']}$ , we have

$$\begin{aligned} \Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \notin \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} \\ = 2^{-n[I(X_1 \wedge X_2|X_3) - \varepsilon']} \end{aligned}$$

by choice, and

$$\begin{aligned} \Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3), \mathbf{X}_3 = \mathbf{x}_3\} &= \frac{\Pr\{K_1 = k, \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}}{\Pr\{\mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_1 \in \mathcal{F}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}} \\ &= \frac{\sum_{j=1}^{N_i(\mathbf{x}_3)} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}}{\sum_{j=1}^{N_i(\mathbf{x}_3)} 2^{n[I(X_1 \wedge X_2|X_3) - \varepsilon']} \Pr\{\mathbf{X}_1 = \mathbf{b}_{i,j,k}(\mathbf{x}_3) | \mathbf{X}_3 = \mathbf{x}_3\}} \\ &= 2^{-n[I(X_1 \wedge X_2|X_3) - \varepsilon']}. \end{aligned}$$



Hence,

$$\begin{aligned} \Pr\{K_1 = k | \mathbf{P}\mathbf{X}_1^t = \mathbf{P}\mathbf{e}_i^t, \mathbf{X}_3 = \mathbf{x}_3\} \\ = 2^{-n[I(X_1 \wedge X_2 | X_3) - \epsilon']} \\ = \Pr\{K_1 = k\}, \end{aligned}$$

where the previous equality follows from (39). Thus,  $K_1$  is independent of  $(\mathbf{X}_3, \mathbf{F})$ , establishing (29). ■

## V. IMPLEMENTATION WITH LDPC CODES

We outline an implementation using LDPC codes (cf. e.g., [21], [30], [34], [31]) of the scheme for the construction of a SK for Model 1 in Section III. As will be indicated below, similar implementations can be applied to Models 2–4 as well.

### A. SK construction

Without any loss of generality, we consider a systematic  $(n, n - m)$  LDPC code  $\mathcal{C}$  with generator matrix  $\mathbf{G} = [\mathbf{I}_{n-m} \ \mathbf{A}]$ , where  $\mathbf{I}_{n-m}$  is an  $(n - m) \times (n - m)$ -identity matrix and  $\mathbf{A}$  is an  $(n - m) \times m$ -matrix. Then, the parity check matrix for  $\mathcal{C}$  is  $\mathbf{P} = [\mathbf{A}^t \ \mathbf{I}_m]$ , where  $\mathbf{I}_m$  is an  $m \times m$ -identity matrix. The first  $n - m$  bits of every codeword in  $\mathcal{C}$ , namely the *information bits*, are pairwise distinct. Further, since the coset with coset leader  $\mathbf{e}_i$ ,  $1 \leq i \leq 2^m$ , must contain the sequence  $\mathbf{b}_i = [\mathbf{0}_{n-m} \ \mathbf{e}_i \mathbf{P}^t]$ , with  $\mathbf{0}_{n-m}$  denoting a sequence of  $n - m$  zeros, the first  $(n - m)$ -bit-segments of the sequences in the coset  $\{\mathbf{b}_i \oplus \mathbf{c}, \mathbf{c} \in \mathcal{C}\}$  are pairwise distinct.

Terminal 1 transmits the syndrome  $\mathbf{P}\mathbf{x}_1^t$ , whereupon terminal 2, knowing  $(\mathbf{x}_2, \mathbf{P}\mathbf{x}_1^t)$ , applies the belief-propagation algorithm described in [19] to estimate  $\hat{\mathbf{x}}_2(1)$ . Since the first  $n - m$  bits of the sequences in each coset are pairwise distinct, these bits can serve as the index of a sequence in its coset. Then, terminal 1 (resp. 2) sets  $K_1$  (resp.  $K_2$ ) as the first  $n - m$  bits of  $\mathbf{x}_1$  (resp.  $\hat{\mathbf{x}}_2(1)$ ).

The same implementation of the SW data compression scheme above holds for Models 2 and 4, too. It can be applied repeatedly also for the successive estimates (20) in Model 3. In Model 3,  $K_{i^*}$  (resp.  $K_i$ ,  $i \neq i^*$ ) is set as the first  $n - m$  bits of  $\mathbf{x}_{i^*}$  (resp.  $\hat{\mathbf{x}}_i(i^*)$ ). It should be noted that the *current complexity of generating regular subsets in Models 2 and 4 poses a hurdle for explicit efficient constructions of a SK and a PK, respectively, for these models.*

### B. Simulation Results

We provide simulation results for the tradeoff between the relative secret key rate (i.e., the difference between the SK capacity and the rate of the generated SK) and the rate of generating unequal SKs at different terminals (corresponding to the bit error rate in SK-matching), when LDPC codes are used for SK construction in Model 1.

For the purpose of comparison, three different LDPC codes were used: (i) a (3, 4)-regular LDPC code; (ii) a (3, 6)-regular LDPC code; and (iii) an irregular LDPC code with degree distribution pair (cf. [19])

$$\begin{aligned} \lambda(x) &= 0.234029x + 0.212425x^2 + 0.146898x^5 \\ &\quad + 0.102840x^6 + 0.303808x^{19}, \\ \rho(x) &= 0.71875x^7 + 0.28125x^8, \end{aligned}$$

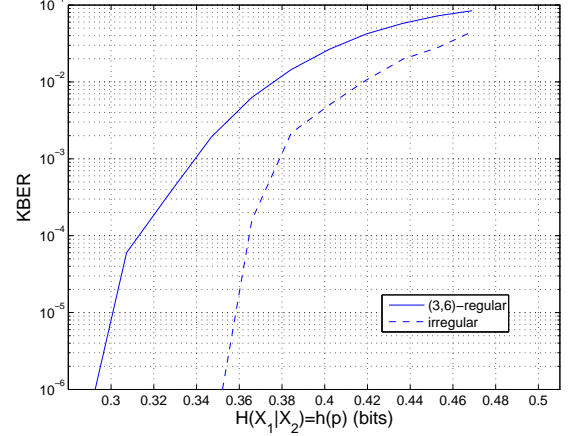


Fig. 1. Simulation results for the (3, 6)-regular and the irregular LDPC codes.

with a common codeword length of  $10^3$  bits, and upto 60 iterations of the belief-propagation algorithm were allowed. Over  $10^3$  blocks were transmitted from terminal 1.

Simulation results are shown in Figures 1 and 2, where conditional entropy (i.e.,  $H(X_1|X_2) = h(p)$ ) is plotted against key bit error rate (KBER). We note that in this simulation SKs are generated at fixed rates that are equal to the rates of the LDPC codes used. Since for Model 1, SK capacity equals  $1 - h(p)$ , the conditional entropy  $h(p)$  serves as an indicator of the gap between SK capacity and the rate of the generated SK.

Figure 1 shows the performance of the (3, 6)-regular and the irregular LDPC codes; Figure 2 shows the performance of the (3, 4)-regular LDPC code. It is seen in both figures that KBER increases with  $h(p)$ . Since SK capacity decreases with increasing  $h(p)$ , an increase of  $h(p)$  narrows the gap between SK capacity and the rate of the generated SK, but raises the likelihood of generating unequal SKs at the two terminals.

It is seen from Figure 1 that the irregular LDPC code outperforms the (3, 6)-regular LDPC code. For instance, for a fixed crossover probability  $p = 0.068$ , say, and  $h(p) \approx 0.3584$ , the KBER for the irregular LDPC code is as low as  $10^{-5}$ , while the KBER for the (3, 6)-regular LDPC code is only about  $4 \times 10^{-3}$ .

## VI. DISCUSSION

We have considered four simple secrecy generation models involving multiple terminals, and propose a new approach for constructing SKs and PKs. This approach is based on Wyner's well-known SW data compression code for sources connected by virtual channels with additive independent noise.

In all the models considered in this paper, the i.i.d. sequences observed at the different terminals possesses the following structure: They can be described in terms of sequences at *pairs of terminals* where each terminal in a pair is connected to the other terminal by a virtual communication channel with additive independent noise.

There are two steps in the SK construction schemes. The first step constitutes SW data compression for the purpose

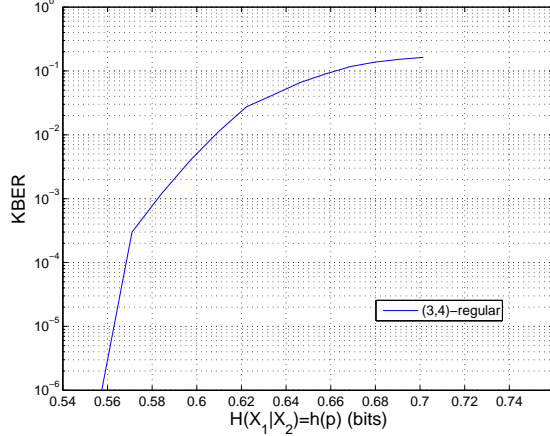


Fig. 2. Simulation results for the (3, 4)-regular LDPC code.

of common randomness generation at the terminals. Although the existence of linear data compression codes with rate arbitrarily close to the SW bound has been long known for *arbitrarily correlated* sources [5], constructions of such linear data compression codes are understood in terms of the cosets of linear error-correction codes for the virtual channel, say  $P_{X_1|X_2}$ , only when this virtual channel is characterized by (independent) additive noise [37]. For instance, when two terminals are connected by a virtual BSC  $P_{X_1|X_2}$ , a linear data compression code, which attains the SW rate  $H(X_1|X_2)$  for terminal 2 to reconstruct the signal at terminal 1, is then provided by a linear channel code which achieves the capacity of the BSC  $P_{X_1|X_2}$ .

When the i.i.d. sequences observed at terminals 1 and 2 are *arbitrarily correlated*, the associated virtual communication channel  $P_{X_1|X_2}$  connecting them is no longer symmetric and corresponds to a virtual channel with input-dependent noise. In this case, while linear codes are no longer rate-optimal for the given channel [10], linear code constructions for a suitably enlarged “semisymmetric” channel that are used for SW data compression [14] could pave the way for devising schemes for SK construction.

The second step in the SK construction schemes involves SK extraction from the previously acquired CR. It has been shown [25] that for the special case of a two-terminal source model, this extraction can be accomplished by means of a linear transformation. However, it is unknown yet whether this holds also for a general source model with more than two terminals.

#### APPENDIX A: PROOF OF PROPOSITION 1

We shall prove (7) here. The proof of (8), which is similar, is omitted. Fix  $\delta > 0$  and consider the set  $T_{[P_X]_\delta}^n$  of sequences in  $\mathcal{X}^n$  which are  $P_X$ -typical with constant  $\delta$  (cf. [6, p. 33]), i.e.,

$$T_{[P_X]_\delta}^n = \{\mathbf{x} \in \mathcal{X}^n : \max_{a \in \mathcal{X}} |P_{\mathbf{x}}(a) - P_X(a)| \leq \delta\}.$$

Since  $T_{[P]_\delta}^n$  is the union of the sets of those types  $\tilde{P}$  of sequences in  $\mathcal{X}^n$  that satisfy

$$\max_{a \in \mathcal{X}} |\tilde{P}(a) - P_X(a)| \leq \delta, \quad (\text{A.1})$$

we have

$$\begin{aligned} & \sum_{\mathbf{x} \in (T_{[P_X]_\delta}^n)^c} P_X^n(\mathbf{x}) \\ &= \sum_{\tilde{P}: \max_{a \in \mathcal{X}} |\tilde{P}(a) - P_X(a)| > \delta} P_X^n(\{\mathbf{x} : P_{\mathbf{x}} = \tilde{P}\}) \\ &\leq (n+1)^{|\mathcal{X}|} \cdot 2^{-n \min_{\tilde{P}: \max_{a \in \mathcal{X}} |\tilde{P}(a) - P_X(a)| > \delta} D(\tilde{P}||P_X)}, \quad (\text{A.2}) \end{aligned}$$

using the fact that  $P_X^n(\{\mathbf{x} : P_{\mathbf{x}} = \tilde{P}\}) \leq 2^{-nD(\tilde{P}||P)}$  (cf. [6, Lemma 2.6]).

Next, by Pinsker's inequality (cf. e.g., [6, p. 58]),

$$\begin{aligned} D(\tilde{P}||P) &\geq \frac{1}{2ln2} \left( \min_{a \in \mathcal{X}} |\tilde{P}(a) - P_X(a)| \right)^2 \\ &\geq \frac{\delta^2}{2ln2}, \quad (\text{A.3}) \end{aligned}$$

with the previous inequality holding for every  $\tilde{P}$  in (A.1). It follows from (A.2) and (A.3) that

$$\sum_{\mathbf{x} \in T_{[P_X]_\delta}^n} P_X^n(\mathbf{x}) \geq 1 - (n+1)^{|\mathcal{X}|} \cdot 2^{-n \frac{\delta^2}{2ln2}} \quad (\text{A.4})$$

for all  $n \geq 1$ .

Finally, observe that

$$T_{[P_X]_\delta}^n \subseteq T_{X,\xi}^n, \quad \text{if } \xi = \delta \left\lceil \sum_{a \in \mathcal{X}} \log \frac{1}{P_X(a)} \right\rceil, \quad (\text{A.5})$$

which is readily seen from the fact that for each  $\mathbf{x} \in \mathcal{X}^n$ ,

$$\begin{aligned} & -\frac{1}{n} \log P_X^n(\mathbf{x}) - H(P_X) \\ &= -\frac{1}{n} \log \left( 2^{-n[H(P_{\mathbf{x}}) + D(P_{\mathbf{x}}||P_X)]} \right) - H(P_X) \\ &= H(P_{\mathbf{x}}) + D(P_{\mathbf{x}}||P_X) - H(P_X) \\ &= H(P_{\mathbf{x}}) - H(P_X) + \sum_{a \in \mathcal{X}} P_{\mathbf{x}}(a) \log \frac{1}{P_X(a)} - H(P_X) \\ &= \sum_{a \in \mathcal{X}} [P_{\mathbf{x}}(a) - P_X(a)] \log \frac{1}{P_X(a)}. \end{aligned}$$

Clearly, (A.4) and (A.5) imply (7).

#### APPENDIX B: PROOF OF PROPOSITION 2

The proof of Proposition 2 relies on the following lemma concerning the average error probability of maximum likelihood decoding.

A sequence  $\mathbf{u} \in \{0, 1\}^n$  is called a *descendent* of a sequence  $\mathbf{v} \in \{0, 1\}^n$  if  $u_i = 1$  implies that  $v_i = 1$ ,  $1 \leq i \leq n$ . A subset  $\Omega \subset \{0, 1\}^n$  is called *quasiadmissible* if the conditions that  $\mathbf{u} \in \Omega$  and  $\mathbf{u}$  is a descendent of  $\mathbf{v}$  together imply that  $\mathbf{v} \in \Omega$ .

**Lemma 2** [22]: If  $\Omega$  is a quasiadmissible subset of  $\{0, 1\}^n$ , then for  $0 \leq p \leq 1$ ,

$$\frac{d\mu_p(\Omega)}{dp} > 0,$$

where

$$\mu_p(\Omega) = \sum_{\mathbf{x} \in \Omega} p^{w_H(\mathbf{x})} (1-p)^{n-w_H(\mathbf{x})},$$

with  $w_H(\mathbf{x})$  denoting the Hamming weight of  $\mathbf{x}$ . ■

For a binary linear code, let  $\mathbf{E}$  denote the set of coset leaders. It is known (cf. [28, Theorem 3.11]) that  $\Omega' = \{0, 1\}^n \setminus \mathbf{E}$  is a quasiamissible subset of  $\{0, 1\}^n$ . If a binary linear code is used on  $\text{BSC}(p)$ , the average error probability of maximum likelihood decoding is given by (cf. [32, Theorem 5.3.3])

$$\mu_p(\Omega') = \sum_{\mathbf{x} \in \Omega'} p^{w_H(\mathbf{x})} (1-p)^{n-w_H(\mathbf{x})}.$$

Lemma 2 implies that if the same binary linear code is used on two binary symmetric channels with different crossover probabilities, say,  $0 < p_1 < p_2 < \frac{1}{2}$ , then the average error probability of maximum likelihood decoding for a  $\text{BSC}(p_1)$  is strictly less than that for a  $\text{BSC}(p_2)$ ; note that a  $\text{BSC}(p_2)$  is a degraded version of a  $\text{BSC}(p_1)$ , being a cascade of the latter and a  $\text{BSC}(\frac{p_2-p_1}{1-2p_1})$ .

Returning to the proof of Proposition 2, it follows from Lemma 1 that for some  $\eta > 0$  and for all  $n$  sufficiently large,

$$\Pr\{\hat{\mathbf{X}}_{j^*}(i^*) \neq \mathbf{X}_{i^*}\} < 2^{-n\eta}.$$

Recall that  $p_{(i^*, j^*)} = \max_{(i, j) \in E(\mathcal{T})} p_{(i, j)}$  and  $(i = i_0, i_1, \dots, i_r = i^*)$  is the path from  $i$  to  $i^*$ . It follows by Lemma 2 that

$$\Pr\{\hat{\mathbf{X}}_i(i_1) \neq \mathbf{X}_{i_1}\} < \Pr\{\hat{\mathbf{X}}_{j^*}(i^*) \neq \mathbf{X}_{i^*}\} < 2^{-n\eta}.$$

Consequently,

$$\begin{aligned} \Pr\{\hat{\mathbf{X}}_i(i_2) \neq \mathbf{X}_{i_2}\} &\leq \Pr\{\hat{\mathbf{X}}_i(i_2) \neq \mathbf{X}_{i_2}, \hat{\mathbf{X}}_i(i_1) \neq \mathbf{X}_{i_1}\} \\ &\quad + \Pr\{\hat{\mathbf{X}}_i(i_2) \neq \mathbf{X}_{i_2}, \hat{\mathbf{X}}_i(i_1) = \mathbf{X}_{i_1}\} \\ &< 2 \cdot 2^{-n\eta}. \end{aligned}$$

Continuing this procedure, we have finally that

$$\Pr\{\hat{\mathbf{X}}_i(i^*) \neq \mathbf{X}_{i^*}\} < r \cdot 2^{-n\eta} < d \cdot 2^{-n\eta}.$$

## REFERENCES

- [1] A. Aaron and B. Girod, "Compression with side information using turbo codes," *Proc. IEEE Data Compression Conference*, pp. 252–261, Snowbird, UT, Apr. 2002.
- [2] R. Ahlswede and I. Csiszár, "Common randomness in information theory and cryptography, Part I: Secret sharing," *IEEE Trans. Inf. Theory*, vol. 39, pp. 1121–1132, July 1993.
- [3] M. Bloch, J. Barros, M. Rodrigues, S.W.M. McLaughlin, "Wireless Information Theoretic Security," *IEEE Trans. on Inf. Theory*, Vol. 54, No. 6, pp. 2515–2534, June, 2008.
- [4] T. P. Coleman, A. H. Lee, M. Médard, and M. Effros, "On some new approaches to practical Slepian-Wolf compression inspired by channel coding," *Proc. IEEE Data Compression Conference*, pp. 282–291, Snowbird, UT, March 2004.
- [5] I. Csiszár, "Linear codes for sources and source networks: Error exponents, universal coding," *IEEE Trans. Inf. Theory*, vol. 28, no. 4, pp. 585–592, July, 1982.
- [6] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Academic, New York, N.Y., 1982.
- [7] I. Csiszár and P. Narayan, "Common randomness and secret key generation with a helper," *IEEE Trans. Inf. Theory*, vol. 46, pp. 344–366, March 2000.
- [8] I. Csiszár and P. Narayan, "Secrecy capacities for multiple terminals," *IEEE Trans. Inf. Theory*, vol. 50, pp. 3047–3061, Dec. 2004.
- [9] P. Elias, "Coding for noisy channels," *IRE Convention Record*, Part 4, pp. 37–46, 1955.
- [10] E. M. Gabidulin, "Bounds for the probability of decoding error when using linear codes over memoryless channels," *Prob. Pered. Inf.*, vol. 3, pp. 55–62, 1967.
- [11] J. Garcia-Frias and Y. Zhao, "Compression of correlated binary sources using turbo codes," *IEEE Commun. Lett.*, vol. 5, pp. 417–419, Oct. 2001.
- [12] J. Garcia-Frias and W. Zhong, "LDPC codes for compression of multi-terminal sources with hidden Markov correlation," *IEEE Commun. Lett.*, vol. 7, no. 3, pp. 115–117, March 2003.
- [13] H.O. Georgii, *Gobbs Measures and Phase Transitions*. de Gruyter, Berlin – New York, 1988.
- [14] D-K. He and E-H. Yang, "On the duality between Slepian-Wolf coding and channel coding," *Proc. Int. Symp. Inf. Theory*, pp. 2546–2550, Seattle, WA, July 2006.
- [15] R. Hu, R. Viswanathan and J. Li, "A new coding scheme for the noisy-channel Slepian-Wolf problem: Separate design and joint decoding," *Proc. Global Commun. Conference*, Dallas, TX, 2004.
- [16] J. Li, Z. Tu, and R. Blum, "Slepian-Wolf coding for nonuniform sources using turbo codes," *Proc. IEEE Data Compression Conference*, pp. 312–321, Snowbird, UT, March 2004.
- [17] C. Lan, A. Liveris, K. Narayanan, Z. Xiong, and C. N. Georgiades, "Slepian-Wolf coding of multiple M-ary sources using LDPC codes," *Proc. IEEE Data Compression Conference*, p. 549, Snowbird, UT, March 2004.
- [18] Y. Liang, H.V. Poor and S. Shamai (Shitz), "Information Theoretic Security," *Foundations and Trends in Communications and Information Theory*, vol. 5, no. 4-5, pp. 355–580, Now Publishers, MA, USA, 2008.
- [19] A. D. Liveris, Z. Xiong, C. N. Georgiades, "Compression of binary sources with side information at the decoding using LDPC codes," *IEEE Commun. Lett.*, vol. 6, pp. 440–442, Oct. 2002.
- [20] A. D. Liveris, C. Lan, K. R. Narayanan, Z. Xiong and C. N. Georgiades, "Slepian-Wolf coding of three binary sources using LDPC codes," *Proc. Int. Symp. Turbo Codes and Related Topics*, Brest, France, Sept. 2003.
- [21] D. J. C. Mackay, "Good error correcting codes based on very sparse matrices," *IEEE Trans. Inf. Theory*, vol. 45, pp. 399–431, Mar. 1999.
- [22] G. A. Margulis, "Probabilistic characteristics of graphs with large connectivity," *Probl. Inf. Trans.* vol. 10, pp. 174–179, Apr. 1974.
- [23] U. M. Maurer, "Secret key agreement by public discussion from common information," *IEEE Trans. Inf. Theory*, vol. 39, pp. 733–742, May 1993.
- [24] P. Mitran and J. Bajcsy, "Turbo source coding: A noise-robust approach to data compression," *IEEE Data Compression Conference*, p. 465, Snowbird, UT, Apr. 2002.
- [25] J. Muramatsu, "Secret key agreement from correlated source outputs using LDPC matrices," *IEICE Trans. Fundamentals*, vol. E89-A, pp. 2036–2046, July, 2006.
- [26] S. Nitinawarat, "Secret key generation for correlated Gaussian sources," *Proc. Allerton Conf. Commun., Control, and Computing*, Monticello, Illinois, pp. 1054–1058, Sept. 2007.
- [27] S. Nitinawarat, "Secret key generation for correlated Gaussian sources," *Proc. IEEE Int. Symp. Inf. Theory*, pp. 702–706, Toronto, Canada, July, 2008.
- [28] W. W. Peterson and E. J. Weldon, *Error-Correcting Codes*, 2nd edition, MIT Press: Cambridge, Mass. 1972.
- [29] S. S. Pradhan and K. Ramchandran, "Distributed source coding using syndromes (DISCUS): Design and construction," *IEEE Trans. Inf. Theory*, vol. 49, pp. 626–643, March 2003.
- [30] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Trans. Inf. Theory*, vol. 47, pp. 599–618, Feb. 2001.
- [31] T. Richardson and R. Urbanke, *Modern Coding Theory*, New York: Cambridge, 2008.
- [32] S. Roman, *Introduction to Coding and Information Theory*, New York: Springer, 1996.
- [33] D. Schonberg, K. Ramchandran and S. S. Pradhan, "Distributed code constructions for the entire Slepian-Wolf rate region for arbitrarily correlated sources," *Proc. IEEE Data Compression Conference*, pp. 292–301 Snowbird, UT, March 2004.
- [34] R. M. Tanner, "A recursive approach to low complexity codes," *IEEE Trans. Inf. Theory*, vol. 27, pp. 533–547, Sept. 1981.
- [35] A. Thangaraj, S. Dihidar, A. R. Calderbank, S. McLaughlin and J. M. Merolla, "Capacity achieving codes for the wiretap channel with applications to quantum key distribution," e-print cs. IT/0411003, 2004.
- [36] R. Wilson, D. Tse and R. Scholtz, "Channel identification: Secret sharing using reciprocity in ultrawideband channels," *IEEE Trans. Inf. Foren. and Security*, vol. 2, pp. 364–375, Sept. 2007.

- [37] A. D. Wyner, "Recent results in the Shannon theory," *IEEE Trans. Inf. Theory*, vol. 20, pp. 2–10, Jan. 1974.
- [38] C. Ye and P. Narayan, "Secret key and private key constructions for simple multiterminal source models," *Proc. IEEE Int. Symp. Inf. Theory*, pp. 2133–2137, Adelaide, Australia, Sept. 2005.
- [39] C. Ye and P. Narayan, "Secret key constructions for simple multiterminal source models," *Proc. Inf. Theory and Applications Workshop*, San Diego, California, Feb. 2006.
- [40] C. Ye, A. Reznik and Y. Shah, "Extracting secrecy from jointly Gaussian random variables," *Proc. Int. Symp. Inf. Theory*, pp. 2593–2597, July 2006.
- [41] C. Ye, S. Mathur, A. Reznik, Y. Shah, W. Trappe and N. Mandayam, "Information theoretic secret key generation from wireless channels," *IEEE Trans. Inf. Foren. and Security*, vol. 5, pp. 240–254, June 2010.
- [42] R. W. Yeung, *Information Theory and Network Coding*, New York: Springer, 2008.